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THE COMPLEXITY OF COMPUTING THE TUTTE POLYNOMIAL ON TRANSVERSAL MATROIDS

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The complexity of computing the Tutte polynomial $T(\mathcal{M},x,y)$ is determined for transversal matroid $\mathcal{M}$ and algebraic numbers x and y. It is shown that for fixed x and y the problem of computing $T(\mathcal{M},x,y)$ for $\mathcal{M}$ a transversal matroid is #P-complete unless the numbers x and y satisfy $(x-1)(y-1)=1$, in which case it is polynomial-time computable. In particular, the problem of counting bases in a transversal matroid, and of counting various types of "matchable" sets of nodes in a bipartite graph, is #P-complete.

1. Introduction

A *transversal system* is defined by triple $(S,I,\mathcal{A})$, where S is the *ground* set, I is the *index* (or *target*), and $\mathcal{A}=\{S_i : i\in I\}$ is a collection of subsets of S indexed by elements of I. We will use the notation $n=|S|$ and $m=|I|$ throughout. A *partial transversal* of $(S,I,\mathcal{A})$ is a subset $A=\{a_1,\dots a_r\}$ of S such that there exist distinct indices $i_1,\dots i_r$ in I — called a *matched index set* associated with A — having $a_j\in S_{i_j}, j=1,\dots,r$. The collection $\mathcal{I}(S,I,\mathcal{A})$ of partial transversals of $(S,I,\mathcal{A})$ comprises the collection of independent sets of a matroid, which is called the *transversal matroid* associated with $(S,I,\mathcal{A})$ and denoted $\mathcal{M}(S,I,\mathcal{A})$. We denote by $\mathcal{T}$ the class of transversal matroids.

The *rank function* for $\mathcal{M}(S,I,\mathcal{A})$ is the function $\rho: 2^S\to\{0,\dots,m\}$ defined by

$$\rho(A)=\max\{|B| : B\in\mathcal{I}(S,I,\mathcal{A}), B\subseteq A\}.$$

The rank function of any set $A\subseteq S$ can be computed in polynomial time as follows: Define the bipartite graph G_A with vertex set $A\cup I$ and edge set $\{(u,v): v\in I, u\in A\cap S_v\}$. Now find the maximum cardinality matching M on the graph G_A. The set of vertices of A which are contained in an edge of M thus forms a maximum cardinality independent set in $\mathcal{M}$ contained in A, with the associated matched index set being the set of vertices of I in M. Finding maximum cardinality matchings can be done in polynomial time, so the rank function is likewise polynomial-time computable.

A *full rank* transversal system $(S,I,\mathcal{A})$ is one which satisfies $\rho(S)=m$, or equivalently, that there exists a partial transversal A whose matched index set is

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all of I. Given any transversal system $(S,I,\mathscr{A})$, one can easily construct a full rank transversal system $(S,I',\mathscr{A}')$ such that $\mathscr{M}(S,I',\mathscr{A}')=\mathscr{M}(S,I,\mathscr{A})$. Simply find a maximum cardinality matching M on the graph G_S defined above, and let I' be the matched index set incident with M. Now define $\mathscr{A}'=\{S_i : i \in I'\}$. From Theorem 3 in [2] it follows that $\mathscr{I}(S,I',\mathscr{A}')=\mathscr{I}(S,I,\mathscr{A})$, and hence $\mathscr{M}(S,I',\mathscr{A}')=\mathscr{M}(S,I,\mathscr{A})$. Since $(S,I',\mathscr{A}')$ can be constructed from $(S,I,\mathscr{A})$ in polynomial time, we can assume where appropriate that a transversal matroid is represented by a full rank transversal system.

The *Tutte polynomial* for matroid $\mathscr{M}$ with associated rank function ρ is the two-variable polynomial $T(x,y)=T(\mathscr{M},x,y)$ defined by

$$T(x,y)=\sum_{A\subseteq S}(x-1)^{\rho(S)-\rho(A)}(y-1)^{|A|-\rho(A)}.$$

The Tutte polynomial can be evaluated over any field, although in this paper we consider primarily evaluations over the complex numbers. As a technical matter, these values need to be further restricted to those which can be arithmetically manipulated in time polynomial in their representation. The most meaningful choice here is the field $\mathbf{A}$ of *algebraic* numbers, that is, complex numbers that can be represented by a finite degree algebraic extension of the rational numbers. We restrict ourselves henceforth to these evaluations. Evaluations of the Tutte polynomial occur throughout combinatorial theory, applying to colorings, flows, acyclic orientations, "scores", and knots (see [7] for an extensive account of this). Two evaluations for transversal matroids arise in the context of processor scheduling reliability [4]. Given full rank transversal system $(S,I,\mathscr{A})$, let the index set represent *tasks* and the ground set *processors*. Each task $i \in I$ can be performed by any processor in S_i, with tasks assigned in one-to-one fashion to available processors. Now suppose the processors fail independently, each with probability q and let $p=1-q$. Then the number $p^m q^{n-m} T(\mathscr{M},1/p,1)$ represents the *independence* probability of the system of processors, that is, the probability that all operating processors can be assigned to tasks. The number $p^m q^{n-m} T(\mathscr{M},1,1/q)$ represents the *span* probability of the system of processors, that is, the probability that all tasks can be assigned to operating processors. More specifically, by setting $p=q=\frac{1}{2}$ and multiplying by 2^n we get that $T(\mathscr{M},2,1)$ counts the *number* of subsets of processors that can all be assigned simultaneously to tasks, and $T(\mathscr{M},1,2)$ counts the number of subsets of processors that can be assigned to simultaneously cover all tasks. These two problems in turn correspond to the graph-theoretic problems of counting, in a bipartite graph $G=(S\cup I, E\subseteq S\times I)$, the number of subsets of S which can be matched to the vertices of I and the number of subsets of S to which all of the vertices of I can be matched, respectively.

A similar use of transversal matroids applies in certain single processor scheduling reliability problems [6]. Other interesting evaluations of the Tutte polynomial can be found in [7].

The evaluation $T(\mathscr{M},1,1)$ is also of importance. $T(\mathscr{M},1,1)$ counts *bases* (maximal independent sets) of $\mathscr{M}$. This corresponds in the the above discussion to the number of subsets of m processors that can be assigned exactly to the set of tasks, or in the graph-theoretic context to the number subsets B of nodes of S for which the graph induced by $B\cup I$ contains a perfect matching. Establishing the complexity of the basis-counting problem for particular classes of matroids has been exceptionally difficult. In fact, except for regular matroids (including graphic and cographic matroids), for which a polynomial algorithm has been known using a generalization of the Kirchoff matrix-tree theorem [8], the complexity of computing $T(\mathcal{M},1,1)$ for any of the major classes of matroids has been unresolved until the results of this paper and [12]. Transversal matroids are also representable over the field of algebraic real numbers, and for any $m \times n$ matrix representation of this matroid, we have that $T(\mathcal{M},1,1)$ counts *matrix bases* — that is, nonsingular column submatrices — of the representing matrix. Finally, the value of $T(\mathcal{M},1,1)$ plays a key role in the efficient approximation of reliability for *shellable independence systems*, of which matroids are an important subclass (see [1], and as it applies to transversal matroids [4]). In particular, the efficient computation of $T(\mathcal{M},1,1)$ for transversal matroids is critical to the success of these methods when applied to the bounding of the independence and span probabilities mentioned above.

We assume the reader is familiar with the notions of Turing machines and NP-complete problems; for an excellent account of this and its relation to #P-completeness, see [5]. Fix input alphabet Σ and denote by Σ^* the corresponding collection of finite strings. We assume Σ^* contains a representation of $\mathbf{Z}_+$. Define the class $\#P$ to consist of those functions $f : \Sigma^* \to \mathbf{Z}_+$ which can be computed by counting accepting computations of some nondeterministic Turing machine of polynomial time complexity. For function $f : \Sigma^* \to \mathbf{Z}_+$ define an *f-oracle Turing machine* to be a Turing machine, together with an additional *input* and *output* tape which, at any time during a computation, can write string σ on the input tape and in one step receive $f(\sigma)$ on the output tape. A function $g : \Sigma^* \to \mathbf{Z}_+$ is *polynomially reducible to* f ($g \propto f$) if there exists a polynomial time complexity f-oracle Turing machine which computes g. A function f is called *#P-complete* if (a) $f \in \#P$, and (b) for each $g \in \#P, g \propto f$. Roughly speaking, the #P-complete problems are those which are polynomially equivalent to the counting problems associated with many NP-complete problems — for example, counting Hamiltonian circuits in a graph. They are therefore at least as hard as NP-complete problems, and so it is unlikely that a polynomial algorithm exists to solve these problems.

The purpose of this paper is to establish the complexity of computing the Tutte polynomial over the class of transversal matroids. In particular, let x_0 and y_0 be two algebraic numbers. For a class of matroids $\mathscr{C}$ define $\tau^0[\mathscr{C},x_0,y_0] : \mathscr{C} \to \mathbf{A}$ to be the function which has as input a matroid $\mathcal{M} \in \mathscr{C}$ and whose output is the value of $T(\mathcal{M},x_0,y_0)$. Now $\tau^0[\mathscr{C},x_0,y_0]$ is technically not in #P, since among other things its output is not restricted to be a natural number. However, by writing the Tutte polynomial in the form

$$T(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} t_{ij}(x-1)^i(y-1)^j$$

where $n = |S|$, $m = \rho(S)$, and

$$t_{ij} = |\{A \subseteq S : \rho(S) - \rho(A) = i, |A| - \rho(A) = j\}|,$$

it follows that computing $T(x_0,y_0)$ for any pair $x_0, y_0 \in \mathbf{A}$ can be done in polynomial time in the size of x_0 and y_0, providing that the values of $t_{ij}, i=0,\dots,m, j=0\dots,n$

are known. The function which sends triples $(\mathcal{M},i,j)$ to t_{ij}, however, is in #P, provided ρ is polynomially computable. As mentioned above this is true for the class of transversal matroids, and so we abuse notation somewhat and say that the Tutte polynomial is also in #P.

The complexity of computing $\tau^0[\mathcal{C},x_0,y_0]$ has been established for the classes $\mathcal{C}$ of graphic matroids, planar graphic matroids, and representable matroids [7], [10], [11], [12]. The purpose of this paper is to extend these complexity results to the class of transversal matroids. The following theorem is the main result of the paper.

Theorem 1. *For the class $\mathcal{T}$ of transversal matroids, and any pair $x_0,y_0 \in \mathbf{A}$, the computation of $\tau^0[\mathcal{T},x_0,y_0]$ is #P-complete unless $(x_0-1)(y_0-1)=1$, in which case it is polynomial.*

We will abbreviate $\tau^0[\mathcal{T},x_0,y_0]$ as $\tau^0[x_0,y_0]$ henceforth.

2. Proof of Theorem 1

A *circuit* of matroid $\mathcal{M}(S,I,\mathcal{A}) \in \mathcal{T}$ is a minimal subset of S which does not lie in $\mathcal{I}(S,I,\mathcal{A})$. The starting point for the complexity results is the following enumeration problem:

NUMBER OF MINIMUM CARDINALITY CIRCUITS IN A TRANSVERSAL MATROID (#MCTM)

Instance: Transversal matroid $\mathcal{M}(S,I,\mathcal{A})$.
Output: Number of minimum cardinality circuits of $\mathcal{M}(S,I,\mathcal{A})$.

Colbourn and Elmallah in [4] prove the following result.

Theorem 2. *#MCTM is #P-complete.*

We also define two additional Tutte polynomial evaluation functions that are useful for intermediate results. For $x_0,y_0 \in \mathbf{A}$ define

$$\tau_x^1[x_0] : \mathcal{T} \to \mathbf{A}[y] \text{ where } \mathcal{M} \mapsto T(\mathcal{M},x_0,y),$$

$$\tau_y^1[y_0] : \mathcal{T} \to \mathbf{A}[x] \text{ where } \mathcal{M} \mapsto T(\mathcal{M},x,y_0).$$

The form of an evaluation of $\tau_x^1[x_0]$ or $\tau_y^1[y_0]$ is a one variable polynomial in y and x, respectively, whose coefficients have size which grows polynomially in the size of x_0 or y_0 and the associated transversal system.

We can immediately state the following corollary to Theorem 2.

Corollary 1. *Computing $\tau_y^1[1]$ is #P-complete; in particular, $\#MCTM \propto \tau_y^1[1]$.*

Proof. $\tau_y^1[1](\mathcal{M}) = \sum_{i=0}^m t_{i0}(x-1)^i$, where t_{i0} is the number of independent sets of cardinality $m-i$ in $\mathcal{M}$. Thus for each $i=0,\dots,n$, $c_i = \binom{n}{m-i} - t_{m-i,0}$ is the number of subsets of cardinality i which are *not* elements of $\mathcal{I}(S,I,\mathcal{A})$. The value of the nonzero c_i of smallest index i is the output required by #MCTM. ∎

We next relate the complexities of computing $\tau_x^1[x_0]$ and $\tau_y^1[y_0]$ to that of computing $\tau^0[x_0,y_0]$. The reductions $\tau^0[x_0,y_0] \propto \tau_x^1[x_0]$ and $\tau^0[x_0,y_0] \propto \tau_y^1[y_0]$ are immediate. To establish the reverse reductions, we first define two intermediate constructions. Fix transversal matroid $\mathcal{M}=\mathcal{M}(S,I,\mathcal{A})$ with associated rank function ρ and Tutte polynomial $T(x,y)=T(\mathcal{M},x,y)$. For nonnegative integer k define the *k-expansion* and *k-augmentation* transversal matroids of $\mathcal{M}$, denoted $\mathcal{M}^{(k)}$ and $\mathcal{M}^{[k]}$, respectively, as follows:

$$\begin{aligned} \mathcal{M}^{(k)} &= \mathcal{M}(S^{(k)}, I^{(k)}, \mathcal{A}^{(k)}), && \text{where} \\ S^{(k)} &= S \\ I^{(k)} &= I \cup K, && \text{and} \\ \mathcal{A}^{(k)} &= \{S_i^{(k)} : i \in I \cup K\}, && \text{with} \\ S_i^{(k)} &= \begin{cases} S_i & i \in I \\ S & i \in K, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}^{[k]} &= \mathcal{M}(S^{[k]}, I^{[k]}, \mathcal{A}^{[k]}), && \text{where} \\ S^{[k]} &= S \cup K \\ I^{[k]} &= I, && \text{and} \\ \mathcal{A}^{[k]} &= \{S_i^{[k]} : i \in I\}, && \text{with} \\ S_i^{[k]} &= S_i \cup K, && i \in I. \end{aligned}$$

The set K in both cases is a set of k elements distinct from S or I. In effect, $\mathcal{M}^{(k)}$ is formed from $\mathcal{M}$ by adding k copies of the set S to $\mathcal{A}$, and $\mathcal{M}^{[k]}$ is formed from $\mathcal{M}$ by adding the elements of K to every set in $\mathcal{A}$. ($\mathcal{M}^{(k)}$ and $M^{[k]}$ are in fact a series of *free/Higgs lifts* and *free extensions*, respectively, of $\mathcal{M}$ on the elements of K [3]). If we let $\rho^{(k)}$ and $\rho^{[k]}$ denote the rank functions for $\mathcal{M}^{(k)}$ and $\mathcal{M}^{[k]}$, respectively, then we have the following result (which can also be derived from [3]).

Lemma 1. *For $A \subseteq S$ and $B \subseteq K$,*

$$\rho^{(k)}(A) = \min\{|A|, \rho(A) + k\},$$
$$\rho^{[k]}(A \cup B) = \min\{m, \rho(A) + |B|\}.$$

Proof. Let $r = \rho^{(k)}(A)$. Let J be an independent set in $\mathcal{M}^{(k)}$ of cardinality r contained in A, with associated matched index set R of cardinality r contained in $I \cup K$. Then the set of elements of J matched with $R \cap I$ must be independent in $\mathcal{M}$ with the remaining indices of R in K, so that $r \le \min\{|A|, \rho(A)+k\}$. Conversely, if J_0 is an independent set of $\mathcal{M}$ of cardinality $\rho(A)$ contained in A, then it is always possible to choose up to k additional elements from $A \setminus J_0$ and match them arbitrarily with the indices of K, so that $r \ge \min\{|A|, \rho(A)+k\}$. This proves the equality for $\rho^{(k)}$.

Now suppose $s = \rho^{[k]}(A \cup B)$, and let J be an independent set in $\mathcal{M}^{[k]}$ of cardinality s contained in $A \cup B$. Then $J \cap S$ must be independent in $\mathcal{M}$, with the remaining elements of J in B, so that $s \le \min\{m, \rho(A)+|B|\}$. Conversely, let J_0

be an independent set of cardinality $\rho(A)$ contained in A, with R_0 the associated matched elements in I. Then the elements of B can be matched arbitrarily with the elements of $I \setminus R_0$, so that $s \geq \min\{m, \rho(A) + |B|\}$. This proves the equality for $\rho^{[k]}$. ∎

The following two results show how the problems of evaluating τ_x^1 and τ_y^1 can be reduced to that of evaluating τ^0, through the use of the k-expansion and k-augmentation matroids.

Proposition 1. *For any pair $x_0, y_0 \in \mathbf{A}$ with $(x_0-1)(y_0-1) \neq 1$, we have*

$$\tau_x^1[x_0] \propto \tau^0[x_0, y_0].$$

Proof. Fix (full rank) transversal matroid $\mathcal{M} = \mathcal{M}(S, I, \mathcal{A})$ for which $\tau_x^1[x_0](\mathcal{M})$ is to be computed. Set $q = n - m$, and for $j = 0, \ldots, q$ define

$$\mathcal{F}^{(j)} = \{A \subseteq S : |A| - \rho(A) = j\}.$$

Note that the $\mathcal{F}^{(j)}$'s partition the subsets of S, since $0 \leq |A| - \rho(A) \leq |S| - \rho(S) = q$ for every $A \subseteq S$. From Lemma 1 it follows that for any $A \in \mathcal{F}^{(j)}$,

$$\rho^{(k)}(A) = \begin{cases} \rho(A) + k & j \geq k \\ |A| & j < k. \end{cases}$$

Now for $k = 0, \ldots, q$ — noting that $\rho^{(k)}(S) = \rho(S) + k$ in this range — the Tutte polynomial for $\mathcal{M}^{(k)}$ at (x_0, y_0) can be written

$$\begin{aligned}
T^{(k)} &= T(\mathcal{M}^{(k)}; x_0, y_0) \\
&= \sum_{A \subseteq S} (x_0 - 1)^{\rho^{(k)}(S) - \rho^{(k)}(A)} (y_0 - 1)^{|A| - \rho^{(k)}(A)} \\
&= \sum_{j=0}^{k-1} \sum_{A \in \mathcal{F}^{(j)}} (x_0 - 1)^{\rho(S) + k - |A|} (y_0 - 1)^{|A| - |A|} \\
&\quad + \sum_{j=k}^{q} \sum_{A \in \mathcal{F}^{(j)}} (x_0 - 1)^{\rho(S) + k - \rho(A) - k} (y_0 - 1)^{|A| - \rho(A) - k} \\
&= \sum_{j=0}^{k-1} \sum_{A \in \mathcal{F}^{(j)}} (x_0 - 1)^{\rho(S) + k - (\rho(A) + j)} (y_0 - 1)^0 \\
&\quad + \sum_{j=k}^{q} \sum_{A \in \mathcal{F}^{(j)}} (x_0 - 1)^{\rho(S) - \rho(A)} (y_0 - 1)^{(\rho(A) + j) - \rho(A) - k} \\
&= \sum_{j=0}^{q} c_j(x_0) \begin{cases} (x_0 - 1)^{k-j} & j < k \\ (y_0 - 1)^{j-k} & j \geq k \end{cases},
\end{aligned}$$

where

$$c_j(x_0) = \sum_{A \in \mathcal{F}(j)} (x_0 - 1)^{\rho(S)-\rho(A)}, j = 0, \ldots, q.$$

The polynomial form of $\tau_x^1[x_0]$ can be written

$$\tau_x^1[x_0](\mathcal{M}) = \sum_{j=0}^{q} c_j(x_0)(y-1)^j,$$

and so the reduction will follow from showing that the $c_j(x_0)$'s can be determined from the values of $T^{(k)}, k = 0, \ldots, q$, in polynomial time. Now the above system of linear equations can be solved in polynomial time, and so all that is required is to show that the solution is unique. This in turn depends on showing that the $(q+1) \times (q+1)$ coefficient matrix U with entries

$$u_{kj} = \begin{cases} (x_0 - 1)^{k-j} & j < k \\ (y_0 - 1)^{j-k} & j \geq k \end{cases}$$

is nonsingular. To establish this, first note that if $y_0 = 1$ then the matrix U is lower triangular with 1's along the diagonal, and so is immediately nonsingular. If $y_0 \neq 1$, then for each $k = 0, \ldots, q$ multiply row 0 by $(y_0 - 1)^{-k}$ and subtract it from row k. This gives matrix U' whose row 0 entries are $1, (y_0 - 1), \ldots, (y_0 - 1)^q$, all of whose other diagonal and above-diagonal entries are 0, and whose subdiagonal entries are all $(x_0 - 1) - (y_0 - 1)^{-1}$. Since $y_0 \neq 1$ and by hypothesis $(x_0 - 1) - (y_0 - 1)^{-1} \neq 0$, then by moving the top row of U' to the bottom we obtain a lower-diagonal matrix whose diagonal entries are nonzero, and this establishes that the matrix U is nonsingular. The proposition follows. ∎

Proposition 2. *For any pair $x_0, y_0 \in \mathbf{A}$ with $(x_0 - 1)(y_0 - 1) \neq 1$, we have*

$$\tau_y^1[y_0] \propto \tau^0[x_0, y_0].$$

Proof. Fix (full rank) transversal matroid $\mathcal{M} = \mathcal{M}(S, I, \mathcal{A})$ for which $\tau_y^1[y_0](\mathcal{M})$ is to be computed. The proof proceeds similarly to that of Proposition 1, by using the values of $T^{[k]} = T(\mathcal{M}^{[k]}, x_0, y_0)$, $k = 0, \ldots, m$, to compute $\tau_y^1[y_0](\mathcal{M})$ in polynomial time. It turns out to be more convenient to work with the matroid $\mathcal{M}^{\langle k \rangle}$ defined by *contracting* the elements of K from $\mathcal{M}^{[k]}$, that is, $\mathcal{M}^{\langle k \rangle}$ is the matroid with ground set S and rank function $\rho^{\langle k \rangle}$ where for all $A \subseteq S$, $\rho^{\langle k \rangle}(A) = \rho^{[k]}(A \cup K) - \rho^{[k]}(K)$. Note that $\mathcal{M}^{\langle k \rangle}$ is not in general a transversal matroid, and is used here only to simplify the presentation. We have that since $k \leq m$ then $\rho^{[k]}(K) = k$, so that $\rho^{\langle k \rangle}(A) = \rho^{[k]}(A \cup K) - k = \min\{m - k, \rho(A)\}$ (and in particular $\rho^{\langle k \rangle}(S) = m - k$).

Defining $T^{\langle k\rangle} = T(\mathcal{M}^{\langle k\rangle}, x_0, y_0)$, it follows that

$$\begin{aligned} T^{[k]} &= \sum_{A\subseteq S}\sum_{B\subseteq K}(x_0-1)^{m-\min\{m,\rho(A)+|B|\}}(y_0-1)^{|A|+|B|-\min\{m,\rho(A)+|B|\}} \\ &= \sum_{B\subseteq K}\sum_{A\subseteq S}(x_0-1)^{(m-|B|)-\min\{m-|B|,\rho(A)\}}(y_0-1)^{|A|-\min\{m-|B|,\rho(A)\}} \\ &= \sum_{b=0}^{k}\binom{k}{b}T^{\langle b\rangle}. \end{aligned}$$

Inverting gives, for $k=0,\dots,m$, the expressions

$$T^{\langle k\rangle} = \sum_{b=0}^{k}\binom{k}{b}(-1)^{k-b}T^{[b]},$$

and so the $T^{\langle k\rangle}$'s can be determined from the $T^{[k]}$'s in polynomial time.

Now for $i=0\dots,m$, define

$$\mathcal{F}^{\langle i\rangle} = \{A\subseteq S : m-\rho(A) = i\}.$$

Again note that the $\mathcal{F}^{\langle i\rangle}$'s partition the subsets of S. From Lemma 1 and the above discussion it follows that for any $A\in\mathcal{F}^{\langle i\rangle}$,

$$\rho^{\langle k\rangle}(A) = \begin{cases} m-k & i\le k \\ \rho(A) & i>k. \end{cases}$$

For $i=0,\dots,m$ define

$$d_i(y_0) = \sum_{A\in\mathcal{F}^{\langle i\rangle}}(y_0-1)^{|A|-\rho(A)},$$

so that the polynomial form of τ_y^1 can be written

$$\tau_y^1[y_0](\mathcal{M}) = \sum_{i=0}^{m} d_i(y_0)(x-1)^i.$$

Expanding $T^{\langle k\rangle}$, $k\ge r$, in the same manner as $T^{(k)}$ was expanded in the proof of Proposition 1, we get that for $k=r,\dots,m$,

$$T^{\langle k\rangle} = \sum_{i=r}^{m} d_i(y_0)\begin{cases}(y_0-1)^{k-i} & i<k \\ (x_0-1)^{i-k} & i\ge k,\end{cases}$$

so that the $d_i(y_0)$'s can be determined from the $T^{\langle k\rangle}$'s (which in turn can be determined from the $T^{[k]}$'s) as in Proposition 1. This completes the proof of the proposition. ∎

Proof of Theorem 1. If $(x_0-1)(y_0-1)=1$ then

$$T(\mathcal{M}, x_0, y_0) = (x_0-1)^{\rho(S)} \sum_{A \subseteq S} (1-y_0)^{|A|} = (x_0-1)^m y_0^n,$$

which can be computed in polynomial time by finding $m=\rho(S)$ as indicated in the first section. If $(x_0-1)(y_0-1)\neq 1$, then using Propositions 1 and 2 we have

$$\tau_y^1[1] \propto \tau^0[x_0, 1] \propto \tau_x^1[x_0] \propto \tau^0[x_0, y_0],$$

since $(1-1)(x_0-1)=0\neq 1$ for any value of x_0. Applying Corollary 1 completes the proof of the theorem. ▮

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